

Unified treatment of hyperfine splittings in $b\bar{b}$, $c\bar{c}$ and $s\bar{s}$.

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We present a relativistic potential model for $b\bar{b}$, $c\bar{c}$ and $s\bar{s}$ using the Cornell potential to extend a two fermion wave equation determined for the study of the hyperfine structure of hydrogen atom and positronium. The hyperfine splitting is obtained by the first perturbation order of the appropriate Breit term. The excellent agreement of the numerical with the experimental results for such a simple potential shows the fundamental importance of relativity in potential quark models. Only three parameters are used for each system. Moreover the same value for the string tension parameter has been used for fitting both the $b\bar{b}$ and $c\bar{c}$ spectra. The numerical results for the $s\bar{s}$ system show the large influence of the hyperfine splitting on the spectrum and are in agreement also with some of the not assigned states.

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INTRODUCTION

Potential models of interacting $q\bar{q}$ systems have a long history and are still a very lively subject of investigation: this is witnessed by the large number of research papers and reviews that keep being published [1], which we refer to for bibliography and exhaustive details on the subject. For systems composed by heavy quarks the starting point is very often the Schrödinger equation with a potential having a Coulomb behavior at the origin and confining at infinity, as the well known Cornell potential; the relativistic corrections together with the spin-orbit and the spin-spin contributions, thought to be determined by one gluon exchange, are taken into account by modifying the potential with the appropriate correcting terms. These are usually treated using a perturbation expansion in the inverse of the heavy quark mass and give the splitting for the states of angular momentum multiplets. Attempts have also been made to overcome the limitations of a potential model due to asymptotic freedom at short distances and to light quark creation: a description of these effects has been tried by means of screened potential softening the Coulomb interaction at the origin and by letting the confining term saturate at infinity. The spin dependent interactions are then modeled by the Breit-Fermi potential with a δ -function centered at the origin, which may yield some difficulties in explaining the hyperfine splittings in the spectra. Although this approximation may be good for heavy mesons, a smearing of the δ -function has been proposed in order to get a better description of the small distance behavior. Recent results [2], however, show that this point is not yet firmly settled. We should also mention that the need of a more complete and consistent relativistic approach to the quarkonium spectrum has sometimes been declared and relativistic models have been actually worked out

[3]. The proposed treatments are often connected with field theory along the lines of the Bethe-Salpeter equation and the spectra of the resulting equations are not of straightforward computation.

In this paper we present an approach to potential models complementary to the lines so far sketched. We use the simplest Cornell potential, focusing instead on a completely covariant formulation and on the fermionic nature of the elementary constituents. Due to the similarity between one-gluon and the one-photon exchanges, we also add a Breit term: it arises from the expansion in the interaction delay of the spatial part of the vector potential and it describes the so called chromomagnetic interaction. Due to its origin, this term must be treated at a first perturbation order. Further improvements of the potential are an important issue which should be developed at a more phenomenological level of the investigation.

Our formulation originates from a wave equation for two relativistic fermions with arbitrary masses [4]. This was obtained in a simple quantum mechanical scheme by coupling two Dirac equations, each one describing a charged fermion in the electromagnetic field generated by the other. In the first of [4] we studied in detail the pure Coulomb problem proving its covariance, the correct Schrödinger limit as well as of the one-particle Dirac limit when one of the masses tends to infinity; we also proved the cyclicity of the relative time that avoids the difficulties of the relative energy excitations. The radial equations were obtained and the two-body Coulomb spectrum was found. In the second paper [4] the hyperfine structure was investigated by adding the vector potential contribution. The spectrum of Positronium including the hyperfine splitting was then calculated. We also got numerical evidence confirming that the spin-spin Breit interaction should be treated at the first perturbation order: including higher orders or even the complete non-perturbative contribution gives back wrong answers,

as expected. Eventually we compared our results with those obtained by QED semi-classical expansions finding an agreement better than up to the fourth power of the fine structure constant.

THE TWO-FERMION WAVE EQUATION WITH CORNELL POTENTIAL AND BREIT TERM

Although we cannot reproduce here the derivation of the two-fermion relativistic wave equation given in [4] we want, however, to stress some of its properties. It is conceptually very simple; the calculations to find the radial system, although rather long, are straightforward and modeled on the scheme of the spherical spinors. The Dirac equations entering the wave equation prescribe, then, the correct form for the interactions according to their tensorial nature: the Coulomb-like term of the Cornell potential is vectorial and thus minimally coupled to

State	Exp	Num
$(1^1s_0) 0^+(0^{--}) \eta_b$	9390.90 ± 2.8	9390.39
$(1^3s_1) 0^-(1^{--}) \Upsilon$	$9460.30 \pm .25$	9466.10
$(1^3p_0) 0^+(0^{++}) \chi_{b0}$	$9859.44 \pm .73$	9857.41
$(1^3p_1) 0^+(1^{++}) \chi_{b1}$	$9892.78 \pm .57$	9886.70
$(1^1p_1) 0^-(1^{+-}) h_b$	9898.60 ± 1.4	9895.35
$(1^3p_2) 0^+(2^{++}) \chi_{b2}$	$9912.21 \pm .57$	9908.14
$(2^1s_0) 0^+(0^{--}) \eta_b$	-	9971.14
$(2^3s_1) 0^-(1^{--}) \Upsilon$	$10023.26 \pm .0003$	10009.04
$(1^3d_1) 0^-(1^{--}) \Upsilon$	-	10143.84
$(1^3d_2) 0^-(2^{--}) \Upsilon_2$	10163.70 ± 1.4	10152.69
$(1^1d_2) 0^+(2^{--}) \eta_{b2}$	-	10154.79
$(1^3d_3) 0^-(3^{--}) \Upsilon_3$	-	10160.91
$(2^3p_0) 0^+(0^{++}) \chi_{b0}$	$10232.50 \pm .0009$	10232.36
$(2^3p_1) 0^+(1^{++}) \chi_{b1}$	$10255.46 \pm .0005$	10256.58
$(2^1p_1) 0^-(1^{+-}) h_b$	-	10263.61
$(2^3p_2) 0^+(2^{++}) \chi_{b2}$	$10268.65 \pm .0007$	10274.26
$(3^1s_0) 0^+(0^{--}) \eta_b$	-	10334.98
$(3^3s_1) 0^-(1^{--}) \Upsilon$	$10355.20 \pm .0005$	10364.52
$(3^3p_0) 0^+(0^{++}) \chi_{b0}$		10534.86
$(3^3p_1) 0^+(1^{++}) \chi_{b1}$	$<10530 \pm .014>_J$	10556.59
$(3^3p_2) 0^+(2^{++}) \chi_{b2}$		10572.44
$(4^3s_1) 0^-(1^{--}) \Upsilon$	$10579.40 \pm .0012$	10655.34
$(5^3s_1) 0^-(1^{--}) \Upsilon$	10876 ± 11	10910.35

TABLE I. The $b\bar{b}$ levels in MeV. In the first column the term symbol, the $I^G(JPC)$ numbers and the particle name. $\sigma=1.111 \text{ GeV}^{-2}$, $\alpha=0.3272$, $m_b=4725.5 \text{ MeV}$. Experimental data from [6].

the energy; the linear term is scalar and therefore coupled to the mass. We recall that only a scalar growing potential is actually confining, while an unbounded vector interaction is not [5]. Hence in our case it is inappropriate to write the Cornell potential just as the sum of the two terms as in the Schrödinger limit where the distinction becomes meaningless. The equation for the $q\bar{q}$ model is obtained by using [4] and coupling the scalar interaction to the fermion masses. We call r_a , q_a the Wigner vectors of spin one given by the spatial parts of relative coordinates and momenta boosted in the frame with vanishing total spatial momentum and we put $r = (r_a r_a)^{1/2}$ (sum over repeated indices). The vector and scalar couplings produce the terms $E + \alpha/r$ and $m + (1/2)\sigma r$. The final form of the two fermions equation contains thus only the three parameters σ , α , m and reads

$$\left[\left(\gamma_{(1)}^0 \gamma_{(1)a} - \gamma_{(2)}^0 \gamma_{(2)a} \right) q_a + \left(\gamma_{(1)}^0 + \gamma_{(2)}^0 \right) \left(m + \frac{1}{2} \sigma r \right) - \left(E + \frac{\alpha}{r} \right) + V_B(r) \right] \Psi(\vec{r}) = 0. \quad (1)$$

In (1) $\gamma_{(i)}$ are the gamma matrices acting in the spinor space of the i -th fermion and

$$V_B(r) = \frac{\alpha}{2r} \gamma_{(1)}^0 \gamma_{(1)a} \gamma_{(2)}^0 \gamma_{(2)b} \left(\delta_{ab} + \frac{r_a r_b}{r^2} \right) \quad (2)$$

denotes the Breit term generating the hyperfine splitting. We found in [4] a very convenient way to evaluate the first order contribution of this term without resorting to eigenfunctions: substitute in (1) $V_B(r)$ by $\varepsilon V_B(r)$ and calculate the first derivative of the eigenvalues with respect to ε in $\varepsilon = 0$ from the numerical solutions of the differential equations. We also proved that the use of eigenfunctions would, obviously, produce the same result.

The radial system is obtained by diagonalizing angular momentum and parity. As in [4] it is formed by four algebraic plus four first order differential equations for each parity. Using the algebraic relations and defining the dimensionless variables Ω , w , s by

$$\sigma = m^2 \Omega^{\frac{3}{2}}, \quad E = m(2 + \Omega w), \quad r = m^{-1} \Omega^{-\frac{1}{2}} s, \quad (3)$$

the radial system for (1), replacing $V_B(r)$ by $\varepsilon V_B(r)$, is

$$\begin{pmatrix} u_1'(s) \\ u_2'(s) \\ u_3'(s) \\ u_4'(s) \end{pmatrix} + \begin{pmatrix} 0 & A_0(s) & -B_0(s) & 0 \\ A_\varepsilon(s) & 1/s & 0 & B_\varepsilon(s) \\ C_\varepsilon(s) & 0 & 2/s & A_\varepsilon(s) \\ 0 & D_\varepsilon(s) & A_0(s) & 1/s \end{pmatrix} \begin{pmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{pmatrix} = 0.$$

Here $A_0 = A_\varepsilon|_{\varepsilon=0}$, $B_0 = B_\varepsilon|_{\varepsilon=0}$ and $u' = du/ds$. Letting $J^2 = j(j+1)$, the coefficients for the even parity are:

$$\begin{aligned} A_\varepsilon(s) &= 0, & B_\varepsilon(s) &= \frac{h(s)}{2} + \frac{\varepsilon\alpha}{s}, \\ C_\varepsilon(s) &= \frac{h(s)}{2} + \frac{2\varepsilon\alpha}{s} + \frac{2J^2}{2\varepsilon\alpha s - s^2 h(s)} + \frac{2s k^2(s)}{4\varepsilon\alpha - s h(s)}, \\ D_\varepsilon(s) &= \frac{2J^2}{s^2 h(s)} - \frac{4\alpha^2 \varepsilon^2 - s^2 h^2(s) + 4s^2 k^2(s)}{4\varepsilon\alpha s - 2s^2 h(s)}, \end{aligned}$$

State	Exp	Num
$(1^1s_0) 0^+(0^{-+}) \eta_c$	2978.40 ± 1.2	2978.26
$(1^3s_1) 0^-(1^{--}) J/\psi$	3096.916 ± 0.11	3097.91
$(1^3p_0) 0^+(0^{++}) \chi_{c0}$	3414.75 ± 0.31	3423.88
$(1^3p_1) 0^+(1^{++}) \chi_{c1}$	3510.66 ± 0.07	3502.83
$(1^1p_1) 0^-(1^{+-}) h_c$	3525.41 ± 0.16	3523.67
$(1^3p_2) 0^+(2^{++}) \chi_{c2}$	3556.20 ± 0.09	3555.84
$(2^1s_0) 0^+(0^{-+}) \eta_c$	3637 ± 4	3619.64
$(2^3s_1) 0^-(1^{--}) \psi$	3686.09 ± 0.04	3692.91
$(1^3d_1) 0^-(1^{--}) \psi$	3772.92 ± 0.35	3808.48
$(1^3d_2) 0^-(2^{--})$	-	3833.62
$(1^1d_2) 0^+(2^{-+})$	-	3839.20
$(1^3d_3) 0^-(3^{--})$	-	3855.18
$(2^3p_0) 0^+(0^{++}) \chi_{c0}$	-	3898.00
$0^+(?^{?+}) \mathbf{X}(3872)$	3871.57 ± 0.25	
$(2^3p_1) 0^+(1^{++}) \chi_{c1}$	-	3961.21
$(2^1p_1) 0^-(1^{+-}) h_c$	-	3977.71
$0^+(?^{?+}) \mathbf{X}(3915)$	3917.4 ± 2.7	
$(2^3p_2) 0^+(2^{++}) \chi_{c2}$	3927 ± 2.6	4003.93
$?^+(?^{??}) \mathbf{X}(3940)$	3942 ± 13	
$(3^1s_0) 0^+(0^{-+}) \eta_c$	-	4064.21
$(3^3s_1) 0^-(1^{--}) \psi$	4039 ± 1	4122.95
$(2^3d_1) 0^-(1^{--}) \psi$	4153 ± 3	4200.51
$(4^3s_1) 0^-(1^{--}) \psi$	4421 ± 4	4479.22

TABLE II. The $c\bar{c}$ levels in MeV. $\sigma=1.111 \text{ GeV}^{-2}$, $\alpha=0.435$, $m_c=1394.5 \text{ MeV}$. Experimental data from [6].

with $h(s) = (2 + \Omega w)/\sqrt{\Omega} + \alpha/s$, $k(s) = (2 + \Omega s)/(2\sqrt{\Omega})$. The coefficients for the odd parity system are:

$$\begin{aligned}
A_\varepsilon(s) &= \frac{2\sqrt{J^2}k(s)}{2\alpha\varepsilon - s h(s)}, \\
B_\varepsilon(s) &= \frac{4\alpha^2\varepsilon^2 - s^2h^2(s) + 4s^2k^2(s)}{4\alpha\varepsilon s - 2s^2h(s)}, \\
C_\varepsilon(s) &= \frac{h(s)}{2} + \frac{2J^2}{2\alpha\varepsilon s - s^2h(s)} + \frac{2\varepsilon\alpha}{s}, \\
D_\varepsilon(s) &= -\frac{h(s)}{2} + \frac{2J^2}{s^2h(s)} - \frac{\alpha\varepsilon}{s}.
\end{aligned}$$

Before presenting the results, a word is in order about the numerical method we have used. The solution of the singular boundary value problem has been obtained by a double shooting method, the spectral condition being the vanishing of the 4×4 determinant of the matching conditions in a crossing point [4]. Padé techniques have been used in order to improve the accuracy of the approximate solutions in the neighborhood of the two singular points, zero and infinity. The integration precision has always

State	$\Delta_B(b\bar{b})$	$\Delta_B(c\bar{c})$	$\Delta_B(s\bar{s})$
$(1^1s_0) 0^+(0^{-+})$	92.31	155.22	296.81
$(1^3s_1) 0^-(1^{--})$	18.09	38.80	94.37
$(1^3p_0) 0^+(0^{++})$	44.30	117.41	297.14
$(1^3p_1) 0^+(1^{++})$	19.98	52.14	127.83
$(1^1p_1) 0^-(1^{+-})$	15.95	43.24	110.77
$(1^3p_2) 0^+(2^{++})$	7.51	21.10	55.93
$(2^3s_1) 0^-(1^{--})$	24.31	60.02	134.22
$(1^3d_1) 0^-(1^{--})$	17.49	49.32	123.85

TABLE III. The Breit correction Δ_B in MeV for some levels of $b\bar{b}$, $c\bar{c}$, $s\bar{s}$.

been kept very high and tested against the stability of the spectral values.

NUMERICAL RESULTS FOR $b\bar{b}$ AND $c\bar{c}$

Let us present the numerical results for the levels of $b\bar{b}$ and $c\bar{c}$. In order to have a test as good as possible of the relevance of the relativistic dynamics in quarkonium

State	Exp	Num
$(1^1s_0) 0^+(0^{-+})$	-	818.12
$(1^3s_1) 0^-(1^{--}) \phi$	1019.455 ± 0.020	1019.44
$(1^3p_0) 0^+(0^{++})$	-	1206.44
$(1^3p_1) 0^+(1^{++}) \mathbf{f}_1(1420)$	1426.4 ± 0.9	1412.84
$(1^1p_1) 0^-(1^{+-})$	-	1458.59
$(1^3p_2) 0^+(2^{++}) \mathbf{f}'_2(1525)$	1525 ± 5	1525.60
$(2^1s_0) 0^+(0^{-+})$	-	1554.68
$(2^3s_1) 0^-(1^{--}) \phi$	1680 ± 20	1698.41
$?^?(1^{--}) \mathbf{X}(1750)$	1753.5 ± 3.8	
$(1^3d_1) 0^-(1^{--})$	-	1776.53
$(1^3d_2) 0^-(2^{--})$	-	1838.72
$(2^3p_0) 0^+(0^{++})$	-	1841.12
$(1^1d_2) 0^+(2^{-+})$	-	1851.44
$(1^3d_3) 0^-(3^{--}) \phi_3(1850)$	1854 ± 7	1880.85
$(2^3p_1) 0^+(1^{++})$	-	1988.38
$(2^1p_1) 0^-(1^{+-})$	-	2021.97
$(2^3p_2) 0^+(2^{++})$	-	2073.15
$(3^1s_0) 0^+(0^{-+})$	-	2099.15
$(3^3s_1) 0^-(1^{--}) \phi$	2175 ± 15	2217.57

TABLE IV. The $s\bar{s}$ levels in MeV. $\sigma=1.34 \text{ GeV}^{-2}$, $\alpha=0.6075$, $m_s=134.27 \text{ MeV}$. Experimental data from [6].

models, we have aimed at choosing the least number of parameters for fitting the experimental data. Flavor independence could be expected for heavy quarks. In fact, doing separate fits for $b\bar{b}$ and $c\bar{c}$ we find that the string tensions turn out to get the same value within the computation precision: we can thus directly assume a single parameter σ for both models so that the total number of parameters is only five. The parameters α , constant in each system, are independent on the energy scale and they are determined by separate fits. However the ratio $\alpha(b\bar{b})/\alpha(c\bar{c}) = 0.752$, numerically found, is very close to the ratio of the two $\alpha_S(\eta_b(1S))/\alpha_S(\eta_c(1S)) = 0.746$ for average values $\Lambda_S = 0.30, 0.33$ GeV for $n_f = 4, 3$ [6].

The spectra show common features, generally shared by all potential models: the states group into doublets of s states and quadruplets of p, d, \dots states. It clearly appears that the results are in very good agreement with experimental data below the threshold of B and D mesons [6] for $b\bar{b}$ and $c\bar{c}$ respectively. Above the threshold the calculated energy of the levels is larger than the experimental one and a softening of the potential growth could make a sensible difference in the agreement with the data for the higher levels. The regularity of the level pattern is however maintained. For the resonance $X(3782)$, having the two possible assignments $J^{PC} = 1^{++}$ and 2^{-+} [6], the model could indicate a χ_{c1} classification. Nothing can be suggested for $X(3915)$ and $X(3940)$, having no accepted quantum numbers.

The situation is simpler in Table I, where there are no unclassified physical states. We point out the good estimate of the recently discovered $\chi_b(3P)$ resonance [6], staying just below the B production threshold. Instead, as expected, the calculated values for $\Upsilon(4^3S_1)$ and $\Upsilon(5^3S_1)$ exceed the experimental values.

We present in Table III the calculated absolute values of the Breit corrections for some states of $b\bar{b}$, $c\bar{c}$ and $s\bar{s}$: they are responsible for the hyperfine splitting and have to be subtracted from the initial pure Cornell potential levels in order to obtain the values given in Tables I, II and IV. As expected, the corrections decrease for increasing values of J and become more and more important for decreasing masses of the component quarks.

A LOOK AT $s\bar{s}$ AND SOME CONCLUSIONS

The good agreement obtained for $b\bar{b}$ and $c\bar{c}$ has urged us to test our model also on the more elusive $s\bar{s}$ system, for which there are few accepted experimental states and many puzzles that cannot be explained by a simple potential model. We believe, however, that these numerical results may be of interest, since the much lighter mass of the s quark highly enhances the relativistic character of the $s\bar{s}$ composite system and the fundamental role of the Breit corrections, giving rise to huge hyperfine splittings. Due to these reasons the string tension σ has not

been given the same value of the previous systems but has been considered a fitting parameter, finding a value larger than in $b\bar{b}$. We report our results in Table IV, where we have also included the unassigned $f_1(1420)$, $X(1750)$, $\phi_3(1850)$ and $\phi(2170)$. Although we cannot have a complete phenomenological confidence in the numerical results, still a fair number of experimental data can be accommodated with a pretty good accuracy. For instance the model could suggest a (1^3d_1) assignment for $X(1750)$.

To conclude we can summarize that the two-fermion relativistic equation describes in a unified way three different types of $q\bar{q}$ systems with the same treatment of the hyperfine splitting as the one successfully applied to positronium. With a very limited number of parameters it provides satisfactory fits for the $q\bar{q}$ levels below the thresholds of lighter meson production and a concrete numerical method to evaluate the effect of the Breit term. It appears thus to be an effective tool in the analysis of two fermion bound states in a given range of energy with a prescribed interaction law.

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